§2-3 5-cycle relation

$M_{0,4}$: moduli of curves of genus $=0$
w/ (ordered) 4 marked pts

$\cong (\mathbb{P}^1; (a,b,c,d))$ $(a,b,c,d) \longmapsto (0,1,\infty,x)$, $x \neq 0,1,\infty$

$\mathbb{P}^1 \setminus \{0,1,\infty\} \ni x$

$M_{0,5}$: : : w/ (ordered) 5 marked pts

$(a,b,c,d,e) \longmapsto (0,1,\infty,x,y)$

$x \neq y$, $x,y \neq 0,1,\infty$

$\cong \left(\mathbb{P}^1 \setminus \{0,1,\infty\}\right)^2 \setminus (\text{diagonal})$

$M_{0,5} \subset \overline{M}_{0,5}$: stable cpt'ation

→ boundary

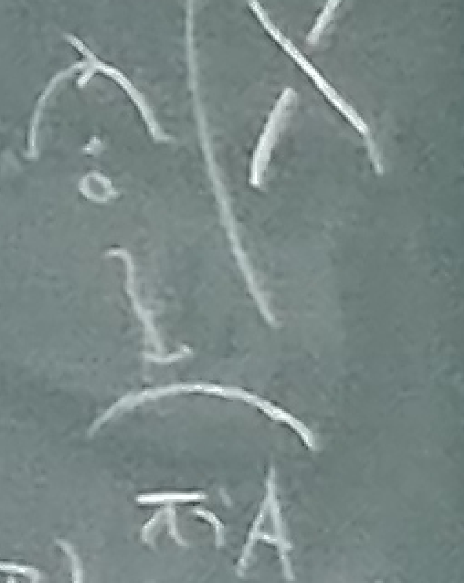
3-cycle rel'n

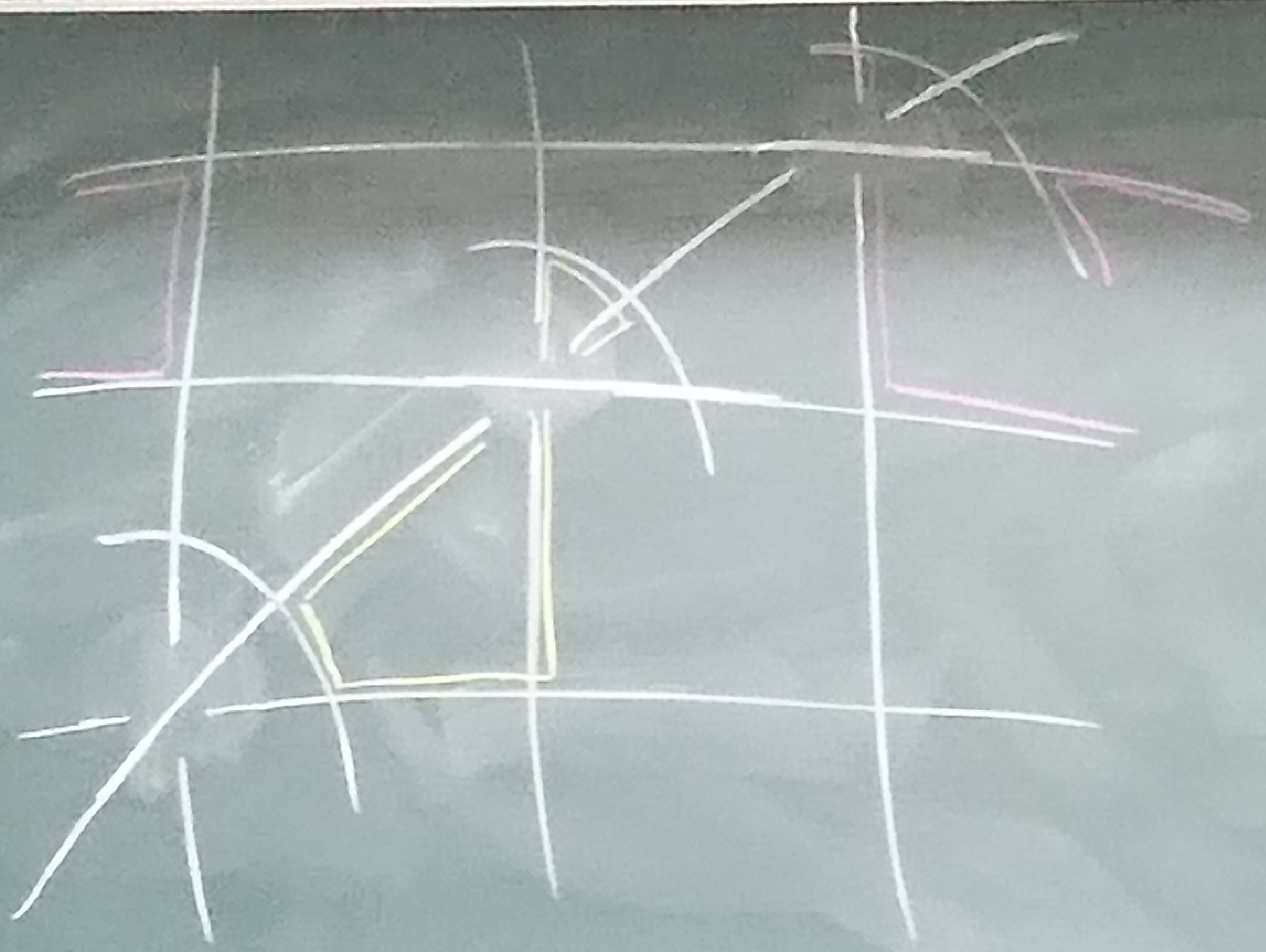
homotopic to 0

(integral lattice model)

R-matrix

$C := (AB)^{-1}$





10 $\mathbb{P}^1$ on boundary

& 12 ⬠s

$$\Longrightarrow \Phi_{KZ}(x_{12}, x_{23})\,\Phi_{KZ}(x_{34}, x_{45})\,\Phi_{KZ}(x_{51}, x_{12})$$

$$\Phi_{KZ}(x_{23}, x_{34})\,\Phi_{KZ}(x_{45}, x_{51}) = 1$$

in $\pi_1^B(M_{0,5}(\mathbb{C}))$ ← Malcev c.

$$\Big(\varprojlim_n \mathbb{C}[P_5]/(\mathrm{aug})^n \quad \text{Malcev compl. of } P_5 \longrightarrow \underline{P_5}(\mathbb{C}) \Big/ \left(\begin{matrix}\text{center}\\ \text{sphere rel'n}\end{matrix}\right)$$

P

center

$= \Big($

$x_{ij} \in P_5$

$i \quad j$

§2-3 5-cycle relation

$M_{0,4}$: moduli of curves of genus $= 0$
w/ (ordered) 4 marked pts

$\cong (\mathbb{P}^1; a, b, c, d) \quad (a,b,c,d) \longmapsto (0,1,\infty,x)$

$x \neq 0,1,\infty$

$M_{0,5}$:

$(a, b, c$

$\cong$

$(\mathbb{P}^1 \setminus \{0,1,\infty$

10 $\mathbb{P}^1$ on boundary

& 12 ⬠ s

$\Rightarrow \Phi_{KZ}(x_{12}, x_{23})\Phi_{KZ}(x_{34}, x_{45})\Phi_{KZ}(x_{51}, x_{12})$

$\Phi_{KZ}(x_{23}, x_{34})\Phi_{KZ}(x_{45}, x_{51}) = 1$

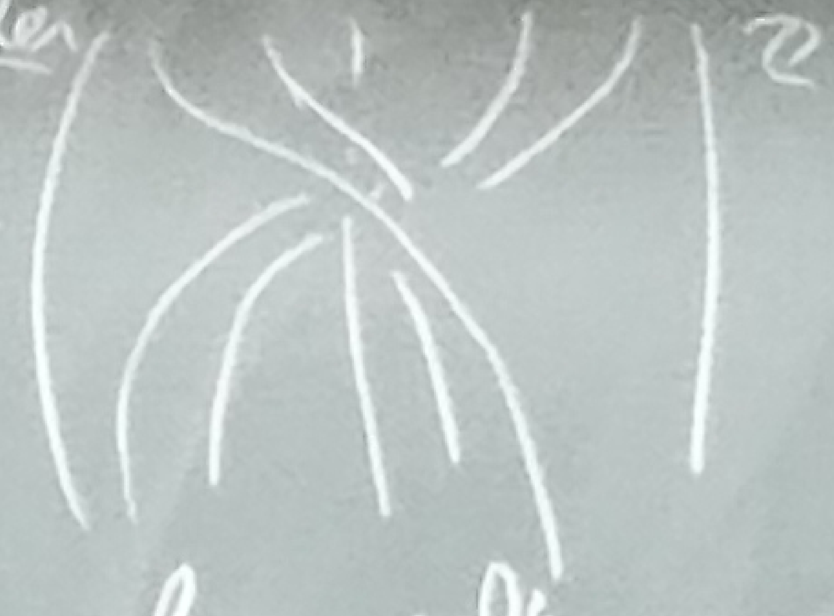
5-cycle relation in $\pi_1^B(M_{0,5}(\mathbb{C}))$ ← Malcev c.

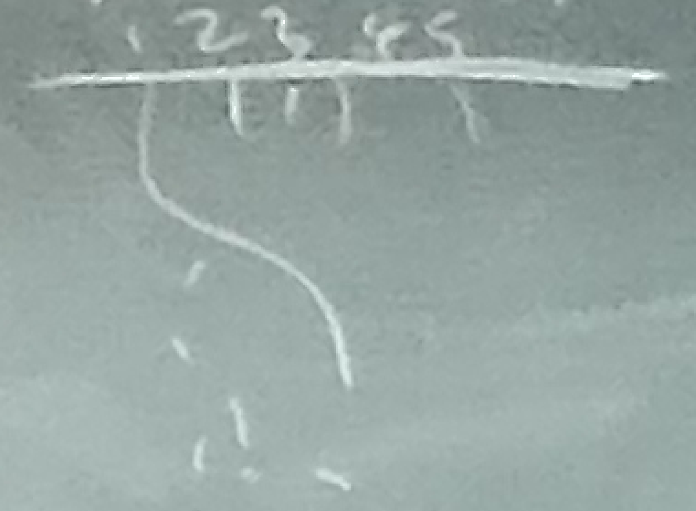
$\|$

$\left(\varprojlim_n \mathbb{C}[P_5]/(\text{aug})^n\right.$ Malcev compl. of P_5 → $\underline{P_5}(\mathbb{C})\Big/\left(\begin{array}{c}\text{center}\\ \text{sphere rel'n}\end{array}\right)$

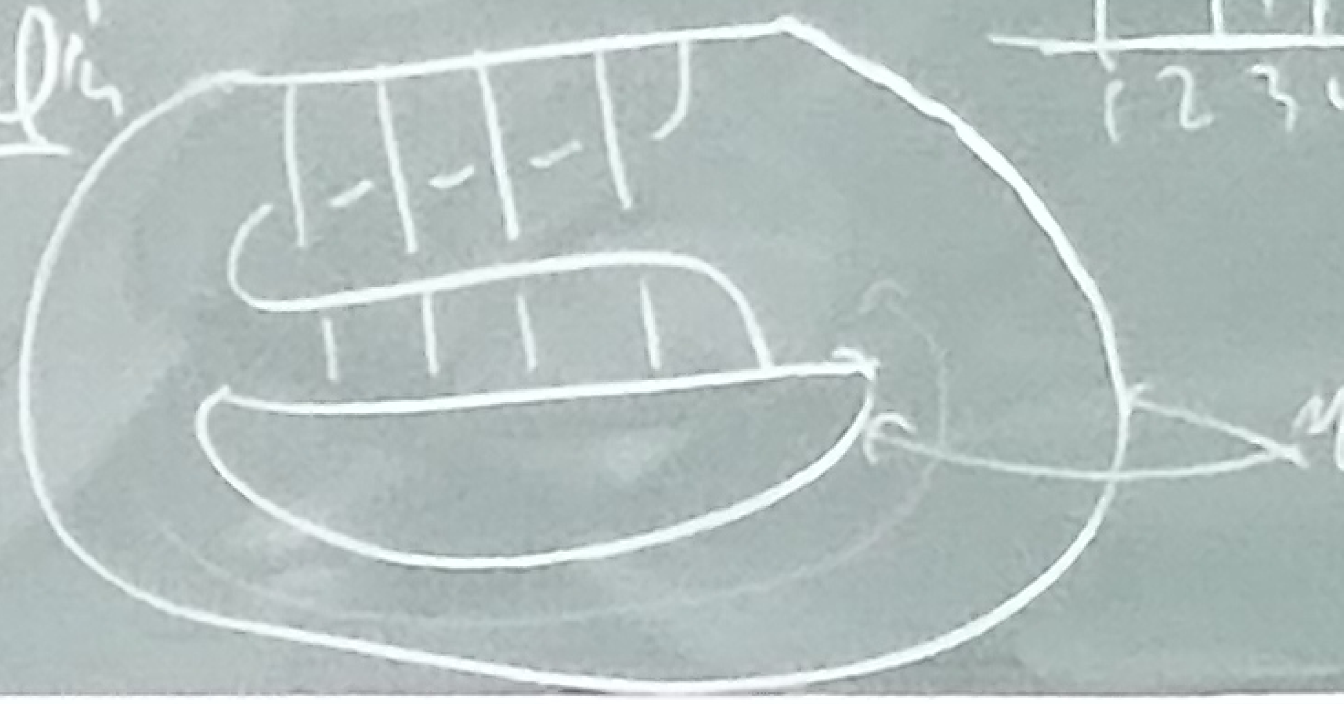
P_5: pure braid gp (i.e. ker$(B_5 \twoheadrightarrow S_5)$) w/ 5 strands

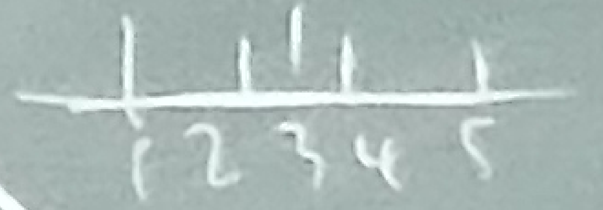
center =





sphere rel'n





spheres

$M_{0,5}$: : : w/ (ordered) 5 marked pts

$(a,b,c,d,e) \longmapsto$

$\|\wr$

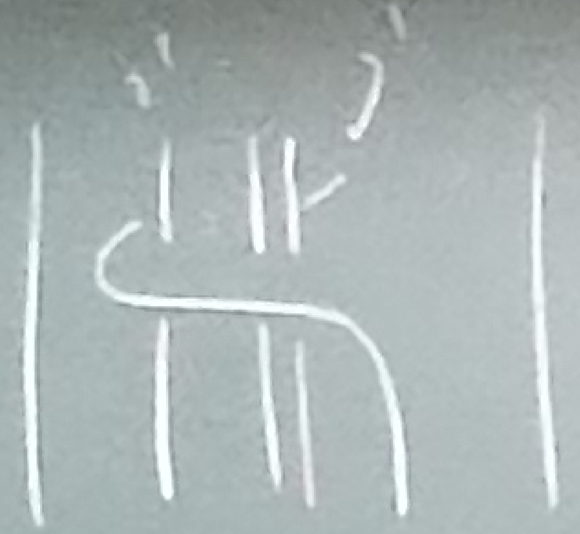
$(\mathbb{P}^1 \setminus \{0,1,\infty\})^2 \setminus \text{diagona}$ ... ∞

stable cpt'ication

boundary

$\mathbb{P}^1(0,1,\infty)$

$x_{ij} \in P_5$

$=$ 

later

$$\begin{tikzcd}
((A\otimes B)\otimes C)\otimes D \arrow[rr, "\Phi_{A,B,C}\otimes \mathrm{id}_D", "\sim"'] \arrow[d, "\Phi_{A\otimes B,C,D}", "\wr"'] & \circlearrowleft & (A\otimes(B\otimes C))\otimes D \arrow[d, "\Phi_{A,B\otimes C,D}", "\wr"'] \\
(A\otimes B)\otimes(C\otimes D) \arrow[dr, "\Phi_{A,B,C\otimes D}"', "\sim"] & & A\otimes((B\otimes C)\otimes D) \arrow[dl, "\mathrm{id}_A\otimes\Phi_{B,C,D}", "\wr"'] \\
& A\otimes(B\otimes(C\otimes D)) &
\end{tikzcd}$$

k : field of char $= 0$

$\underline{GT}(k) := \{(\lambda, f$

R-matrix

pro-algebraic

cf. $\widehat{GT} := \{($

Rem

Th

rel'ns

regu

$\ldots \mid \mid \otimes D$

$F_{A, B \otimes C, D}$

$\otimes C \mid \otimes D)$

$\otimes \underline{F}_{B,C,D}$

$\otimes D))$

Malcev compl. of F_2 ← free gp of rk = 2

k : field of char = 0

$\underline{GT}(k) := \{ (\lambda, f) \in k^\times \times \underline{F}_2(k) \mid f \in \underline{F}_2'(k),\ \lambda, f$ satisfy 2-, 3-, 5-cycle rel'ns $\}$

λ: R-matrix, f: associator

pro-algebraic Grothendieck-Teichmüller gp

cf. $\widehat{GT} := \{ (\lambda, f) \in \hat{\mathbb{Z}}^\times \times \hat{F}_2 \mid f \in \hat{F}_2',\ \lambda, f$ satisfy 2-, 3-, 5-cycle rel'ns $\}$

$\hat{F}_2$: prof. compl. of F_2

profinite Grothendieck-Teichmüller gp

Rem Th (Furusho)
association rel'ns
imply regularized
double shuffle rel'n

Th (Furusho) (only pro-alg. version)
pentagon rel'n implies hexagon & 2-cycl
(5-cycle) (3-cycle) rel'ns

§3. K-theory relation

§3-1 Tannakian in

$U := \mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}$

- Tannakian cat. of

fiber functor
corresponding to $\vec{01}$
tang

- Tannakian cat. of vector bdles on $\mathbb{P}^1$
unipotent

k : field of

$\underline{GT}(k) :=$

pro-alge

2-cycl rel'ns

§3. K-theory relation (sketch)

§3-1 Tannakian interpretation

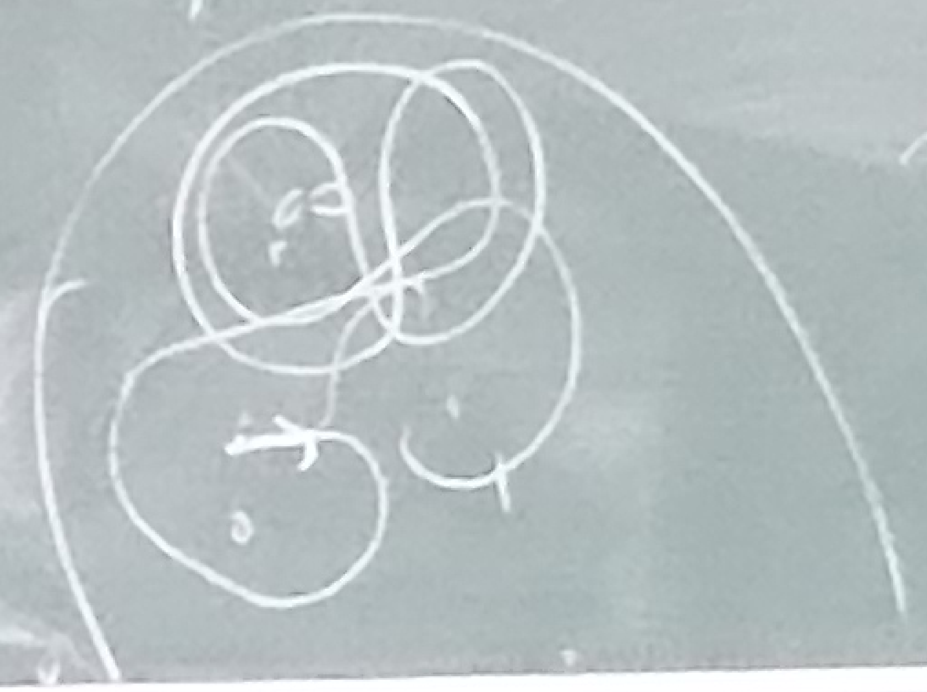
$U := \mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}$ $\quad z \in U(\mathbb{C})$

• Tannakian cat. of unipotent ($\mathbb{Q}$-) local systems on $U(\mathbb{C})$

$$\pi_1^B(U(\mathbb{C}); \vec{01}, z) = \mathrm{Isom}^{\otimes}(\vec{01}, z)$$

fiber function corresponding to z / $\vec{01}$

tangential base pt



Betti prounipotent fundamental groupoid of $U(\mathbb{C})$

pro-alg. gp $/\mathbb{Q}$

$\infty\}$

· Tannakian cat. of vector bdles on $\mathbb{P}^1$
w/ unipotent integrable connections
w/ log poles along $\{0,1,\infty\}$

$\underbrace{\text{fiber functor}}_{\text{corr. to } \vec{01}, z}$ $\pi_1^{dR}(U/\mathbb{Q}; \vec{01}, z) = \mathrm{Isom}^{\otimes}(\vec{01}, z)$

pro-alg sp/$\mathbb{Q}$

de Rham prounipotent fundamental groupoid
of $U/\mathbb{Q}$

choose B_z: topological path from $\vec{01}$ ...

$\rightsquigarrow B_z \in \pi_1^{B}(U$ ...

$H^1(U, \mathcal{O}_U) = 0 \rightsquigarrow {}^{\forall}\pi_1^{dR}$ ...

$\rightsquigarrow \exists! \, dR_z$ ...

$\vec{10} \rightsquigarrow (dR_{\vec{10}})^{-1} \circ B_{\vec{10}} \rightsquigarrow$ MZV's
Φ_{KZ}

$\exists$ p-adic analogue

§3. K-theory rel...

§3-1 Tannakia...

...ion

...ry $\{0,1,\infty\}$

$(\vec{01}, z)$

...orld

$U/\mathbb{Q}$

choose B_z: topological path from $\vec{01}$ to z on $U(\mathbb{C})$

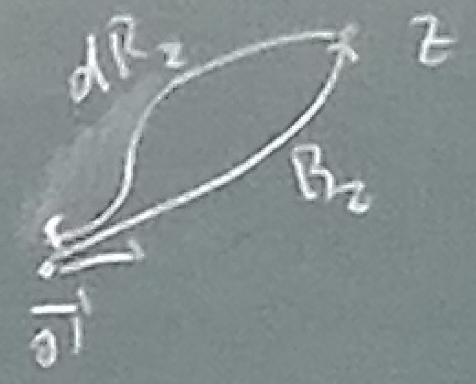
$\rightsquigarrow B_z \in \pi_1^B(U(\mathbb{C}); \vec{01}, z)$

$H^1(U, \mathcal{O}_U) = 0 \rightsquigarrow {}^{\forall}\pi_1^{dR}(U/\mathbb{Q}, \vec{01})$-torsor is trivial

$\rightsquigarrow \exists!\, dR_z \in \pi_1^{dR}(U/\mathbb{Q}; \vec{01}, z)$

Th (Chen)

$$\mathbb{C} \otimes_{\mathbb{Q}} \pi_1^B(U(\mathbb{C}), \vec{01}, z) \cong \mathbb{C} \otimes_{\mathbb{Q}} \pi_1^{dR}(U/\mathbb{Q}; \vec{01}, z)$$



$$(dR_z)^{-1} \circ B_z \in \pi_1^{dR}(U/\mathbb{Q}; \vec{01})(\mathbb{C})$$

$$\hookrightarrow \mathbb{C}\langle\langle A, B\rangle\rangle$$

$$G_0(z) = \sum_w (-1)^{\mathrm{depth}(w)} Li_w(z)$$

(B_z branch of Li_w) multiple polylog corr. to w

§3. K-theory relation (sketch)

§3-1 Tannakian interpretation

$U := \mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}$ $z \in U(\mathbb{C})$

- Tannakian cat. of unipotent ($\mathbb{Q}$-) local systems on $U(\mathbb{C})$

$$\pi_1^B(U(\mathbb{C}); \vec{01}, z) = \mathrm{Isom}^{\otimes}(\vec{01}, z)$$

Betti prounipotent ... groupoid